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A FUNCTIONAL EQUATION IN THE KINETIC THEORY OF GASES.

BY T. H. GRONWALL.

Let ξ, η, ζ and ξ_1, η_1, ζ_1 be the velocity components of two spherical molecules at the instant of their collision, α, β, γ the direction cosines of their center line at this instant, and finally ξ', η', ζ' and ξ_1', η_1', ζ_1' the velocity components after the collision. Writing

$$W = \alpha(\xi_1 - \xi) + \beta(\eta_1 - \eta) + \gamma(\zeta_1 - \zeta),$$

we then have

$$\begin{aligned} \xi' &= \xi + \alpha W, & \eta' &= \eta + \beta W, & \zeta' &= \zeta + \gamma W, \\ \xi_1' &= \xi_1 - \alpha W, & \eta_1' &= \eta_1 - \beta W, & \zeta_1' &= \zeta_1 - \gamma W, \end{aligned}$$

nd a complete system of invariants for this linear transformation is given y the four expressions

$$\xi + \xi_1,\ \eta + \eta_1,\ \zeta + \zeta_1,\ \xi^2 + \eta^2 + \zeta^2 + \xi_1^2 + \eta_1^2 + \zeta_1^2.$$

Denoting by $\varphi(\xi, \eta, \zeta)$ the logarithm of the function defining the distribution of velocities, it may be shown that $\varphi(\xi, \eta, \zeta) + \varphi(\xi_1, \eta_1, \zeta_1)$ is invariant under the linear transformation in question,* and consequently

$$(1) \quad \varphi(\xi, \eta, \zeta) + \varphi(\xi_1, \eta_1, \zeta_1) \\ = f(\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1, \xi^2 + \eta^2 + \zeta^2 + \xi_1^2 + \eta_1^2 + \zeta_1^2).$$

Assuming the existence of all six partial derivatives of the second order of φ, it is readily seen that the general solution of this functional equation is†

$$(2) \qquad \varphi(\xi, \eta, \zeta) = a + b_1\xi + b_2\eta + b_3\zeta + c(\xi^2 + \eta^2 + \zeta^2)$$

with constant coefficients.

It is the purpose of the present note to prove this result without any other assumption than that of the continuity of φ for all finite values of the variables.

From (1) subtract the equation obtained by setting all the variables equal to zero; with the notations

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* See, for a proof involving a minimum of assumptions, Hilbert, "Begründung der kinetischen Gastheorie," Math. Annalen, vol. 72, reprinted in his Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, pp. 267–282.

† Boltzmann, Vorlesungen über Gastheorie, vol. 1, pp. 128–131.

$$\begin{aligned}\varphi_1(\xi, \eta, \zeta) &= \varphi(\xi, \eta, \zeta) - \varphi(0, 0, 0) = \varphi(\xi, \eta, \zeta) - a,\\ f_1(\xi + \xi_1, \cdots) &= f(\xi + \xi_1, \cdots) - f(0, 0, 0, 0)\end{aligned} \tag{3}$$

we obtain

$$\begin{aligned}&\varphi_1(\xi, \eta, \zeta) + \varphi_1(\xi_1, \eta_1, \zeta_1)\\ &\qquad = f_1(\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1, \xi^2 + \eta^2 + \zeta^2 + \xi_1^2 + \eta_1^2 + \zeta_1^2),\end{aligned} \tag{4}$$

$$\varphi_1(0, 0, 0) = 0, \quad f_1(0, 0, 0, 0) = 0. \tag{5}$$

Writing (4) for the two sets of arguments $\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1$ and 0, 0, 0, we find with the aid of (5)

$$\begin{aligned}&\varphi_1(\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1)\\ &\qquad = f_1(\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1, (\xi + \xi_1)^2 + (\eta + \eta_1)^2 + (\zeta + \zeta_1)^2).\end{aligned}$$

In order to make the fourth argument in f_1 coincide with that in (4), we assume that

$$\xi\xi_1 + \eta\eta_1 + \zeta\zeta_1 = 0,$$

and under this condition we have

$$\varphi_1(\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1) = \varphi_1(\xi, \eta, \zeta) + \varphi_1(\xi_1, \eta_1, \zeta_1).$$

Our condition is fulfilled by making $\xi_1 = \eta = \zeta = 0$, whence

$$\varphi_1(\xi, \eta_1, \zeta_1) = \varphi_1(\xi, 0, 0) + \varphi_1(0, \eta_1, \zeta_1),$$

and also by making $\xi = \xi_1 = \eta_1 = \zeta = 0$, which gives

$$\varphi_1(0, \eta, \zeta_1) = \varphi_1(0, \eta, 0) + \varphi_1(0, 0, \zeta_1).$$

Replacing η_1 and ζ_1 by η and ζ in the last two equations, we therefore obtain

$$\varphi_1(\xi, \eta, \zeta) = \varphi_1(\xi, 0, 0) + \varphi_1(0, \eta, 0) + \varphi_1(0, 0, \zeta). \tag{6}$$

Combining (4) and (6), we see at once that

$$\begin{aligned}&f_1(\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1, \xi^2 + \eta^2 + \zeta^2 + \xi_1^2 + \eta_1^2 + \zeta_1^2)\\ &\qquad = f_1(\xi + \xi_1, 0, 0, \xi^2 + \xi_1^2) + f_1(0, \eta + \eta_1, 0, \eta^2 + \eta_1^2)\\ &\qquad\qquad + f_1(0, 0, \zeta + \zeta_1, \zeta^2 + \zeta_1^2).\end{aligned} \tag{7}$$

We introduce the notations

$$\begin{aligned}\xi + \xi_1 = x, \quad \eta + \eta_1 = y, \quad \zeta + \zeta_1 = z,\\ \xi^2 + \xi_1^2 = u, \quad \eta^2 + \eta_1^2 = v, \quad \zeta^2 + \zeta_1^2 = w;\end{aligned} \tag{8}$$

these six quantities are independent variables which, however, to insure the reality of ξ, ξ_1, etc., must satisfy the inequalities

$$x^2 \leqq 2u, \quad y^2 \leqq 2v, \quad z^2 \leqq 2w. \tag{9}$$

Now (7) may be written in the form

$$f_1(x, y, z, u + v + w) = f(x, u) + g(y, v) + h(z, w), \tag{10}$$

where f, g and h are continuous in the region (9) and vanish for zero values of the variables.

Permutating u and v, which is permissible if x^2 and y^2 are both less than or equal to the smaller of $2u$ and $2v$, we find

$$f(x, u) + g(y, v) = f(x, v) + g(y, u), \tag{11}$$

and on making $y = 0$, $v = v_0 =$ const., the last equation gives

$$f(x, u) = f(x) + \rho(u) \tag{12}$$

for $x^2 \leqq 2u$ and $x^2 \leqq 2v_0$. Here, $\rho(u) = g(0, u)$ is continuous for all $u \geqq 0$ and vanishes for $u = 0$, and since (12) may be written $f(x) = f(x, u) - \rho(u)$, it is clear that $f(x)$ is continuous for all finite values of x, and vanishes for $x = 0$. Similarly we obtain $g(y, v) = g(y) + \sigma(v)$, where $g(0) = \sigma(0) = 0$, and substituting in (11) we get $\sigma(v) - \rho(v) = \sigma(u) - \rho(u)$, or making $u = 0$, $\sigma(v) = \rho(v)$. We thus finally have

$$\begin{aligned} f(x, u) &= f(x) + \rho(u), \\ g(y, v) &= g(y) + \rho(v), \\ h(z, w) &= h(z) + \rho(w), \end{aligned} \tag{13}$$

where f, g and h are continuous for all and ρ for positive values of the variable, and all four functions vanish for zero values of their arguments. Substituting in (10) and making $x = y = z = w = 0$, we obtain an equation of the form $\rho_1(u + v) = \rho(u) + \rho(v)$, and making $v = 0$ we see that $\rho_1(u) = \rho(u)$, so that finally

$$\rho(u + v) = \rho(u) + \rho(v). \tag{14}$$

This is the classical functional equation of Cauchy, who showed in his Analyse algébrique that every continuous solution is of the form

$$\rho(u) = cu,$$

where c is a constant.

Consequently, by (8) and (10)

$$\begin{aligned} &f_1(\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1, \xi^2 + \eta^2 + \zeta^2 + \xi_1^2 + \eta_1^2 + \zeta_1^2) \\ &\quad = f(\xi + \xi_1) + g(\eta + \eta_1) + h(\zeta + \zeta_1) + c(\xi^2 + \eta^2 + \zeta^2 + \xi_1^2 + \eta_1^2 + \zeta_1^2), \end{aligned}$$

and on writing

$$\varphi_2(\xi, \eta, \zeta) = \varphi_1(\xi, \eta, \zeta) - c(\xi^2 + \eta^2 + \zeta^2), \tag{15}$$

equations (4) and (6) become

$$\varphi_2(\xi, \eta, \zeta) + \varphi_2(\xi_1, \eta_1, \zeta_1) = f(\xi + \xi_1) + g(\eta + \eta_1) + h(\zeta + \zeta_1), \tag{4'}$$

$$\varphi_2(\xi, \eta, \zeta) = \varphi_2(\xi, 0, 0) + \varphi_2(0, \eta, 0) + \varphi_2(0, 0, \zeta). \tag{6'}$$

In (4′), make $\eta = \zeta = \xi_1 = \eta_1 = \zeta_1 = 0$, whence $\varphi_2(\xi, 0, 0) = f(\xi)$, and then $\eta = \zeta = \eta_1 = \zeta_1 = 0$, which gives

$$f(\xi + \xi_1) = f(\xi) + f(\xi_1),$$

every continuous solution of which is of the form $f(\xi) = b_1\xi$, where b_1 is a constant. Similarly $\varphi_2(0, \eta, 0) = g(\eta) = b_2\eta$ and $\varphi_2(0, 0, \zeta) = h(\zeta) = b_3\zeta$, so that, by (6′),

$$\varphi_2(\xi, \eta, \zeta) = b_1\xi + b_2\eta + b_3\zeta, \tag{16}$$

and the combination of (16), (15) and (3) finally gives the expression (2) for the most general continuous solution of our functional equation.

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